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Closure on "Some comments on recent generalizations of Cattaneo–Mindlin" by J. Jäger

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I have recently replied to a similar letter by Dr. Jäger for the Journal of Applied Mechanics, and I hope this will not become my permanent occupation! However, in the interest of brevity, I will make every possible effort not to repeat the same discussion here.

In the present letter, Dr. Jäger starts with the rather abrupt statement that the subject of my papers is a theory which "was previously published by Jäger". It seems that this problem is bound to receive independent overlapping contributions as it was at the time of Cattaneo (1938) and Mindlin (1949): [1] I confirm that I did not know Dr. Jäger's papers at the time I wrote (Ciavarella, 1998a,b,c, 1999), and until April 1998. Nevertheless, I would like to make it clear that, *with one exception*, there is only a superficial overlap between our papers, as they proceed by different routes, and particularly in the early papers, Dr. Jäger does not give importance to the inequalities of Coulomb's law. Also, the scope of my papers is in general more application oriented. I will therefore reply to just few aspects that deserve further consideration, referring to my previous reply and leaving it to the most careful and interested reader (if any!) who may want to take the time to do such a detailed analysis, to compare the papers line by line.

My interest in the Cattaneo problem started while I was in Oxford working with Prof. D.A. Hills' group on fretting fatigue. At the end of 1996, Prof. Hills was working numerically on the effects of wear in fretting, looking for how large could the effect of the change of geometry be in the evolution of an initially Hertzian contact subjected to oscillating tangential forces and constant normal load. Moved by this interest, I wrote the integral equations for the plane contact and I noticed that in the derivation of the Cattaneo theory in plane problems (see e.g. Hills et al., 1993, Section 4.3) we are just one step away from the general solution: in fact, observing that there is no need to write down explicitly any particular geometry in order to obtain the solution to Cattaneo's problem as a superposition of normal contact solutions. This, together with the proofs that the inequalities also translate, was the 'core' of Ciavarella (1998a). Vice versa, the implications in terms of the original interest in wear were published later (Ciavarella and Hills, 1999) with the interesting result that while wear proceeds in the microslip regions, it never enters the original stick regions; the outcome is that the contact under the conditions described above was demonstrated to proceed towards the limiting state of complete contact in the original stick region.

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[1] Prof. K.L. Johnson told me in June 1998 that R.D. Mindlin had confessed to him several years before that he had lost most of his interest in this problem after discovering Cattaneo's earlier results.

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Turning to the *exception*, I think that looking a posteriori at the papers, only Ciavarella (1998a) has elements in common with one of Dr. Jäger's papers, i.e. the one submitted and published later by Dr. Jäger in J. Tribology (earlier papers did not address this problem so specifically – as we assume that Dr. Jäger does not publish the same material twice!). The circumstance that my motivation was principally the investigation of strength of contacts and fretting fatigue, rather than the pure solution of the contact problem, is witnessed by other publications where the solution of Cattaneo's problem is only the basic ingredient rather than the main objective. The reader interested in applications may find more interest in those papers (Ciavarella, 1999; Ciavarella and Demelio, 1999; Ciavarella and Hills, 1999; Ciavarella et al., 1998, 1999a,b, 2000) than in the purely theoretical ones.

Motivated by further generalizations of these findings in practical applications not idealized by plane contact, I found that the same conclusions applied without significant difference in the fully three-dimensional case *if Poisson's ratio of both contacting materials is zero*, as was reported in (Ciavarella, 1998c), while in the general case the analysis is more complex: indeed, after retrieving two not widely known papers by Cattaneo (1947b,c), I identified the correct framework to approach the general problem. In these papers Cattaneo had basically all the necessary elements for the generalization of his result to the general geometric case, which he did not develop probably because he had in mind as principal application the Hertzian geometry (and indeed, his study of the axisymmetric fourth order profile is only meant as an improvement of the Hertzian theory for truly spherical bodies). The small additional work needed for the discussion of the general axisymmetric case was included in Ciavarella (1999) and is also briefly summarized below. However, the focus of (Ciavarella, 1999) is not on the Cattaneo problem, but rather on practical implications for the induced stress field: the very interesting result was found that the rounded flat indentation gives a generally less severe stress field than the corresponding Hertzian case with same contact area and load, unless the rounded part is very small, with obvious implications for the design of mechanical joints, or foundations.

Turning back to April 1998, I asked many scientists (contacted directly or indirectly) for reprints in connection with a review paper I was writing with Prof. J.R. Barber (Barber and Ciavarella, 1999), including among the others Dr. Jäger. As a result, we had a discussion with him (I even invited him to visit me in Oxford on which occasion I gave him most of my papers on the subject including some preprints and notes). Finally, in the review paper Prof. J.R. Barber and I wrote on a section of that paper dedicated to the frictional problems, literally:

"Cattaneo's results were extended to other loading scenarios by Mindlin and Deresiewicz (1953). A significant generalization of these results has recently been discovered independently by Jäger (1998) [2] and Ciavarella (1998a)..."; "Results for more general loading scenarios have been established by Jäger (1998)..."; "With this caveat, the results of (Ciavarella, 1998c) can probably be automatically extended to the general loading case using the arguments of Jäger (1998). The resulting "Ciavarella–Jäger theorem" would be a very powerful tool in the understanding of frictional quasi-static and impact problems for half-spaces of similar materials."

With this, I think my objective and open attitude towards Dr. Jäger's results is evident.

Turning back to the specific points of Dr. Jäger's letters, I have only a few remarks, subdivided by sections in the same ordering as in Dr. Jäger's letter:

(1) Jäger's theory for elastic friction: I do not agree with the brief summary of the generalized Cattaneo's theory Dr. Jäger self-promotes under his name, for the simple reason that Dr. Jäger mentions an arbitrary reduced version of Coulomb's law for which the only requirement is "that the tangential traction must be

[2] We cited it even if it was at the time just a preprint, later published as J. Jäger, 1998. A new principle in Contact Mechanics. J. Tribology, 120: 677–685.

proportional to the normal pressure''. Therefore, this section is incomplete and the original papers are to be preferred.

(2) Axisymmetric contact: Dr. Jäger derives some alternative notation and some comparative tables between his papers and mine, some minor details on the derivative of the pressure distribution on which I do not have much to say (although I do not understand why he calls the radius $r = b$ as the ''border of the stick area'' while it is clear that it is simply the border of the flat area with the rounded part of the indenter profile). Further, I do not agree with his statement immediately after Eq. (9) where he writes that ''In linear elasticity, solutions can linearly be superposed, and it is not necessary to perform any calculations for the solution in this case''. We are here discussing a contact problem, which is governed by equalities and inequalities, and as such it is not a linear problem (due to the inequalities). Therefore, it is not a general principle that solutions of contact problems can be superposed, unless we fix the contact areas, ''freezing'' the inequalities (and therefore non-linearities). However, this permits us only to compute the pressure distribution but not the relationship between contact area and load.

My main concern about the general geometry of contact is that *not even the equalities* are trivially satisfied. In fact, in the case where we do not neglect the effect of Poisson's ratio, it is not obvious a priori that a solution satisfying the equalities alone exists at all. In contrast to normal contact, where pressure and displacements have only one component, in the tangential direction there are two components active in general. Moreover, given a condition on displacements, i.e. for example our condition $u_x = \delta_x$, $u_y = \delta_y$ in the stick area (we cannot assume $\delta_y = 0$ from the outset), it is not possible to guarantee in general that a solution with just one component of shear q_x (in the direction of the resultant force Q_x) exists at all, as a self-equilibrated distribution of shear q_y could in principle also exist. [3]

In Ciavarella (1999), Eqs. (26) and (27) it is found that the equality conditions on displacements in the stick area can be reduced to [4]

$$\Delta V(x,y) = h_1 + h_2, \quad (x,y) \in S_{\text{stick}}, \tag{1}$$

$$\frac{\partial^2}{\partial x\, \partial y} V(x,y) = -\delta_y/\gamma, \quad (x,y) \in S_{\text{stick}}, \tag{2}$$

where Δ is the 2D-Laplacian operator and $V(x,y)$ is a potential function defined by

$$V(x,y) = \frac{2\pi}{A} \int\int_S R q_x(x', y') \mathrm{d}x' \mathrm{d}y', \tag{3}$$

where A is the composite compliance of the bodies (Ciavarella, 1999, Eq. (2)), R is the distance between field (x,y) and integration point (x',y'), and finally h_1, h_2 are constants defined from the algebraic equation

$$\delta_x = h_1 + (1+\gamma)h_2. \tag{4}$$

The solution of Eq. (1) *alone* can be obtained considering the Cattaneo superposition for q_x and obtaining an integral equation in the stick area in terms of a *corrective* unknown part only $q_x^*(r')/f$, such that (Ciavarella, 1999, Eq. (35))

$$\frac{2\pi}{A} \int\int_{S_{\text{stick}}} \frac{q_x^*(r')/f}{R} \mathrm{d}S = u_z(r) - \frac{h_1 + h_2}{f}, \quad r \in S_{\text{stick}} \tag{5}$$

i.e. the corrective part can be found as the normal contact pressure of the actual problem for a *reduced* indentation, for which the solution evidently exists.

[3] Note that even in the case that symmetry dictates $\delta_y = 0$ from the outset, the uncertainty on the self-equilibrated distribution q_y persists.

[4] When S_{stick} is not simply connected, the constants h_1, h_2 are defined for each connected region.

Therefore, the solution of Eqs. (1) and (2) exists *only if* the solution of Eq. (1) with q_x alone, gives a potential $V(x,y)$ which satisfies *also* (2), i.e. is a *second-degree polynomial* in x and y in S_{stick}. This condition is always satisfied for axisymmetric problems. In fact, in this case the shear distribution q_x, and hence also the potential function V, are functions of r only, so that Eq. (1) reads

$$\Delta V(r) = \frac{\mathrm{d}^2 V(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}V(r)}{\mathrm{d}r} = h_1 + h_2 \tag{6}$$

which has a general solution (neglecting logarithmic terms) $V(r) = ((h_1 + h_2)/4)r^2 + c$.

More general cases of exact solutions are possible, as long as V is a second-degree polynomial in x and y. The most relevant are Hertzian elliptical contacts (as proved explicitly although indirectly by Cattaneo (1938)) and plane problems for which there is no dependence on y and

$$\Delta V(x) = \frac{\mathrm{d}^2 V(x)}{\mathrm{d}x^2} = h_1 + h_2 \tag{7}$$

which obviously has a general solution $V(x) = ((h_1 + h_2)/2)x^2 + c$. Other particular cases of such exact solutions may exist with a more general shape of stick area, but the question is difficult to answer, [5] and of limited practical interest. In fact, given a certain shape of the contact and stick area, the surface profiles needed to satisfy this condition will have a very special geometry and very special values of the normal loading level.

(3) Plane contact: Limiting our attention to the points raised to which I have not yet replied, I do not understand the basis for Dr. Jäger to write in the first sentence that my Eqs. (18) and (19) are "better expressed" by his Eq. (9), and why I should necessarily use his notation. On the contrary, I agree his Eq. (22) is more general than my Eq. (A9) of Ciavarella (1998b). Finally, I do not see any improvements in his rewriting of my remarks on p. 2356 of Ciavarella (1998a).

Closure of the "Closure": One aspect that is certainly unique to Dr. Jäger's papers is the generalized loading conditions, which I suggested (Ciavarella, 1998c) to call "Mindlin problem" as it is Mindlin to first solve this problem for the Hertzian case (1953), whereas the purely sequential loading case should correctly be attributed to Cattaneo (1938, 1947b,c) rather than to Mindlin (1949). Not only the two 1947 papers by Cattaneo (1947b,c) have been unfortunately neglected by the vast audience, but also interesting studies on rolling (Cattaneo, 1939, 1946a,b, 1947a) and on impact (Cattaneo, 1938–1939), which, being an area of interest for Dr. Jäger, I strongly recommend to read.

Turning back to the present matter, rather than the areas of possible previous overlap, I look with interest to the areas of further developments based on the original parts of each others' contributions. For example, as I said in the review paper (Barber and Ciavarella, 1999), it may be interesting to extend my results for the general geometric case in the view of Dr. Jäger's results for the "Mindlin problem". Also, it may be interesting to generalize my results regarding the presence of bulk strains in the contacting materials (Ciavarella et al., 2000) and about wear in the microslip areas (Ciavarella and Hills, 1999), and in general apply the "Mindlin problem" to problems of fretting. Finally, I look forward to reading about the recent generalization of Dr. Jäger to thin layers, although I am sceptical that they may contain rigorous proofs for the inequalities.

Michele CiavarellaBari,
18 October, 1999

[5] I have exchanged few letters in 1998 with Dr. V.I. Fabrikant who has given me a preprint of a paper (probably by now in print) where the problem is further assessed.

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